8 PROJECT COMENIUS DYNAMAT

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FINDING GEOMETRIC PATTERNS AS A GAME OF DYNAMIC EXPLORATIONS

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Abstract

The paper deals with the inquiry based style of learning as applied to traditional and open geometry problems by means of dynamic geometry software. The so called what-if strategy *(i.e. exploring what happens if the formulation of the original problem varies) is demonstrated in the context of a well know problem from the geometry textbook – to find the locus of the midpoints of the segments joining a fixed point within a circle with the points of that circle. After making a dynamic construction for the locus the students are offered additional tools in support of the rigorous proof. The exploration game continues with varying the initial conditions of the problem (e.g. replacing the circle by other figures and the midpoint with a point dividing the segment in a fixed ratio). Then the well known problem of finding the locus of the centers of the equilateral triangles inscribed in an equilateral triangle is considered together with its ambitious generalization, viz. to find the locus of the centers of the regular m-gons inscribed in a regular n-gon ($m \leq n$). The process of generalization leading to open problems is considered together with the construction of appropriate dynamic tools for explorations. It is the very process rather than the description of the results which is of primary interest since it illustrates how the atmosphere around the working mathematicians could be transferred into a class setting. The expectation is that some teachers and students would gain motivation in attacking the considered open problems themselves.*

Keywords

Inquiry based learning, dynamic geometry software, loci related problems, what-if strategy

INTRODUCTION

Many interesting geometric problems deal with finding a locus ─ the set of points satisfying a particular condition. The traditional problems on loci are limited to finding simple curves. Language based computers environments allow for much more sophisticated explorations (Sendov, Sendova, 1995). While the computer language offers a vast spectrum of expressive means, enabling the user to enlighten the finest details of his thought, it is often found to be a great obstacle for the math teachers. Thus, the inquiry based learning in mathematics has been recently promoted within a number of European educational projects (DALEST, *Meeting in Mathematics*, *Math2Earth*, *InnoMathEd*, *Fibonacci*, DynaMAT) by means of dynamic geometry software offering direct manipulation of geometric objects (Christou et al., 2007, Georgiev et al., 2008, Bianco and Ulm, 2010, Baptist and Raab, 2013, Andersen et al., 2010).

In this paper we shall demonstrate how the inquiry based style of learning could be applied in the context of traditional and open geometry problems.

LOOKING AT THE CLASSICS WITH A DYNAMIC EYE

A very important component of the inqury-based mathematics learning is the *what-if* strategy, i.e. to explore what will happen if we vary the formulation of the original problem. Let us illustrate this strategy in the context of a well know problem from the geometry textbooks:

A traditional geometry problem: *What is the locus of the midpoints **M** of the segments joining a fixed point **P** within a circle with the points of that circle?*

To solve this problem by means of dynamic geometry software (say *GeoGebra*) the students study the behaviour of the midpoint under question while moving the endpoint of the segment on the circle along it (Fig. 1a):

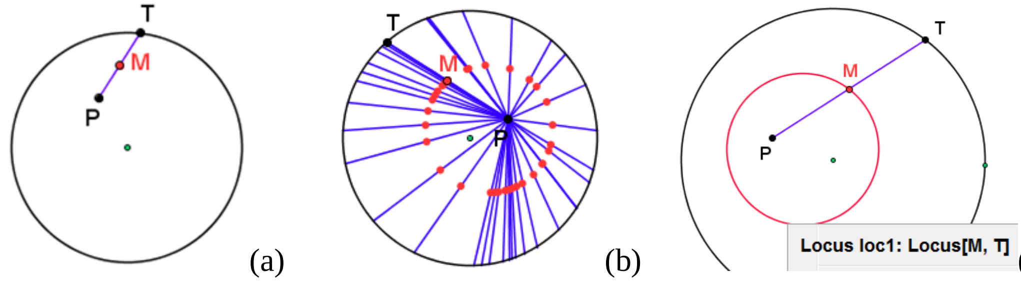

Fig. 1 The first steps of the exploration

They could strengthen their conjectures about the shape of locus by following the the trace of the midpoint's path (possibly with the segment) (Fig. 1b) and finaly chek experimentally their conjectures by constructing the locus of the midpoint by the inbuilt tool (Fig. 1c).

The game is not over, however. It is time to ask some *What-if* questions, e.g. *What if **M** is not the midpoint, but divides the segment at a fixed ratio? What if **P** is outside of the circle?.*

The typical conjecture of the students is that in this case the locus would look like a more general curve of a second degree, e.g. an ellipse. The teacher guides the explorations by suggesting to make the ratio a variable **e** in which **M** is dividing the segment (i.e. to create a slider in our case) (Fig. 2a and 2b):

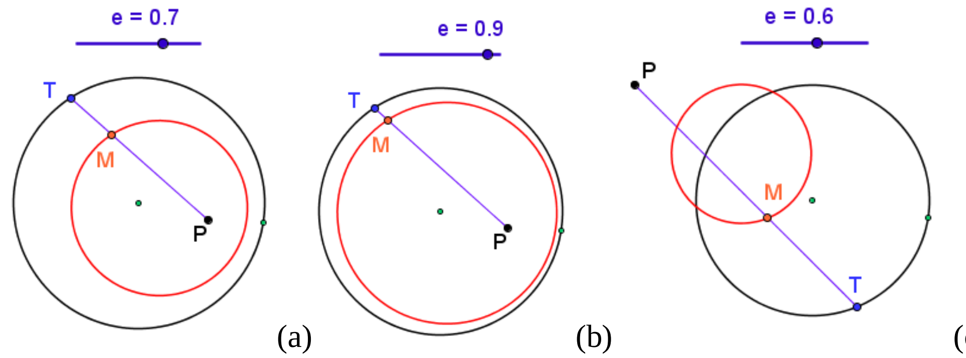

Fig. 2 Changing the ratio of the division

The students are genuinely surprised to find that the locus remains a circle. A further idea arises — to explore the situation when the point **P** is outside of the circle (Fig. 2c) — a circular shape once again! Then a new idea is suggested bringing an interesting effect — to trace the segment for a point outside the circle:

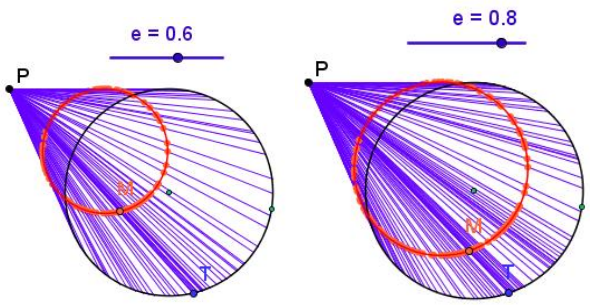

Fig. 3 Changing the position of point P

Now is the time for the teacher to raise students' suspition - could they be absolutely sure that **M** describes a circle? Couldn't it be in fact an ellipse which is very close to a circle…

One way to verify their conjecture (still experimentally) is to construct 3 points (**H**, **I**, **J**) on the locus, pass a circle through them and check if this circle concides with the locus. Another way which could help them prove the conjecture rigorously is to observe some interesting properties of the construction enriched with some auxiliary elements (Fig.4):

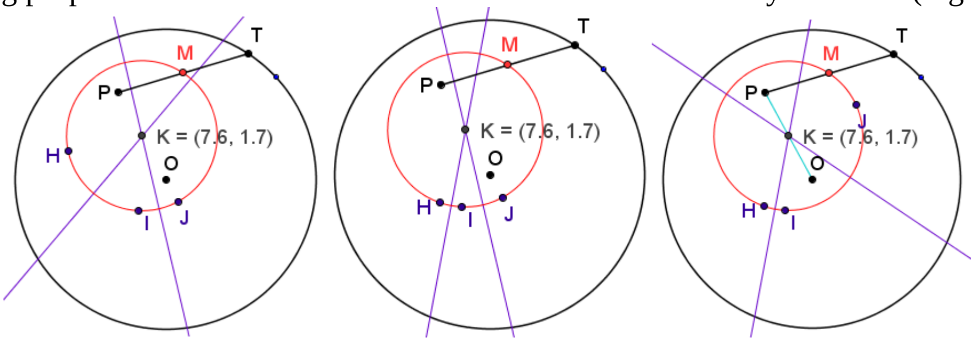

Fig. 4 Auxiliary construction in support of the rigorous proof

It is easy for the students to see that that the center **K** of the locus keeps its coordinates the same. In addition the segments **KJ**, **KI** and **KH** have equal legth which shows experimentally (but **with a greater degree of conviction**) that the locus is a circle. Furthermore, they would notice that **K** is the midpoint of the segment **PO**, where **O** is the center of the original circle. Now they are ready to prove rigorously that the locus is a circle with a center the midpoint **K** of the segment **PO** and a radius ─ half of the radius of the given circle.

The exploration game can continue with replacing the circle by a square, a triangle, an arbitrary regular polygon, a curve of their own choice.

If the students have studied *dilation* (in the Bulgarian curriculum it is introduced a year after the first occurrence of *loci*) they could use it to solve the problems but it is very appropriate for them to get used to generalizing their findings. Applying the *What-if* strategy could cultivate an exploratory spirit in mathematics classes - the students are encouraged to explore interesting partial cases, to generalize relatively simple problems in various directions, and even to attack and generalize challenging problems of Olympic level (Atanasova, 2011).

FROM A WELL KNOWN PROBLEM TO AN OPEN ONE

Here we demonstrate a process which is typical for the working mathematicians – we generalise a well-known problem, then we attack it with tools we believe are the most appropriate for the purpose (in our case with dynamic constructions we have specially designed in a *step-by-step refinement and enrichment* spirit). We systemize our explorations and reflect on the ideas we get. It is the very process that will be of our primary interest rather than the description of the results. In addition, we expect some teachers and students to get motivated in attacking some of the open problems themselves.

A well-known problem:
Find the locus of the centers of the equilateral triangles inscribed in an equilateral triangle.

An ambitions generalization of this problem could be formulated as follows.

An ambitious generalization:
*Find the locus of the centers of the regular **m**-gons inscribed in a regular **n**-gon,* $m \leq n$.

Further below we shall write **(*m*;*n*)** to denote the construction of a regular ***m***-gon inscribed in a regular ***n***-gon. Note that we are not even sure for which ***m*** and ***n*** the **(*m*;*n*)** constructions are possible. Let us start our *attack* with a more modest problem, dealing with the case **(3;*n*)** for ***n*** = 3, 4, …

The first attack – the (3;*n*) case:
*Find the locus of the centers of the equilateral triangles inscribed in a regular **n**-gon.*

A primitive (hand-made) dynamic model

We construct an equilateral triangle two of whose vertices are on the ***n***-gon and move the third one so as to get an inscribed triangle**.** To get the flavor of the dynamic construction to be then generalized it is natural to start with the simplest case (***n*=3)**, and proceed in what could be called a *hand-made* model (Fig.5):

- We select two arbitrary points **M** and **N** on different sides of the given (the *blue*) triangle.
- Then we construct an equilateral (*red*) triangle with a side **MN.**
- Next we move **N** (keeping **M** at irs current position) so that the red triangle becomes inscribed in the blue one. The center of the red triangle is a point of the locus sought.
- Now we repeat the above proces for a new position of **M.**

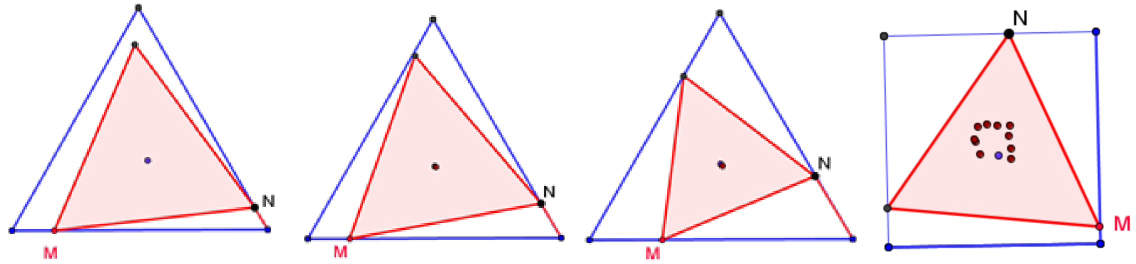

Fig. 5 The hand made **(3;3)** and **(3;4)** dynamic models

Thus, using consecutive positions of point **M** we get an approximate idea about the locus **–** in the **(3;3)** case the centers seem to coincide (or are at least close enough)… If we apply a similar procedure for the **(3;4)** case the centers appear to be on a square. But inscribing the triangle *by hand* is a time-consuming method (still better than constructing on a paper and considering just one possibly misleading case due to imprecision (Pehova, 2011).

To automize the construction let us take a better look at the **(3;3)** construction. It is natural to conjecture that in this case the locus is a single point coinciding with the center of the given triangle (Fig.6).

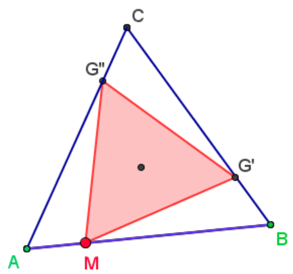

Fig. 6 The **(3;3)** dynamic construction

The congruence of the triangles **AMG"** and **BG'M** implies **AM**=**BG'**. Therefore, we can use in this particular case a dynamic construction based on the congruence.

An automized dynamic model for (3;n) constructions

There are various methods of creating automized models for the **(3;3)** constructions. Here is one of them:

- We construct a point **M** on the contour of a regular **3**-gon`(the triangle **ABC**)

- We construct the image **G'** of **M** under a rotation of 120° about the center of **ABC**
- We construct the image **G''** of **G'** under a rotation of 120° about the center of **ABC**
- We connect the points **M, G'** and **G''** in a triangle.

For **n**>3 we can proceed as follows:

- We construct a point **M** on the contour of a regular **n**-gon.
- Then we construct the image of the **n**-gon under a rotation **ρ** of 60° about **M**.
- We construct their interestion point **F.** (It will be another vertex of the equilateral triangle whose first vertex is **M**, and which is inscribed in the ***n***-gon.)
- Then we construct the thrid vertex as the pre-image **F'** of **F**.
- We connect **M**, **F'** and **F** to get the equilateral triangle inscribed in the ***n***-gon.

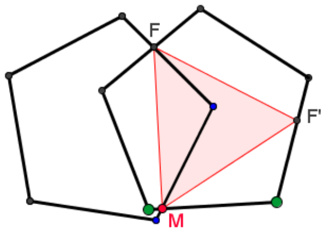

Fig. 7 Constructing a (**3;5**) dynamic model

Here are some snap-shots of the trace the triangle's center in the **(3;4)** construction leaves during the movement of the inscribed triangle:

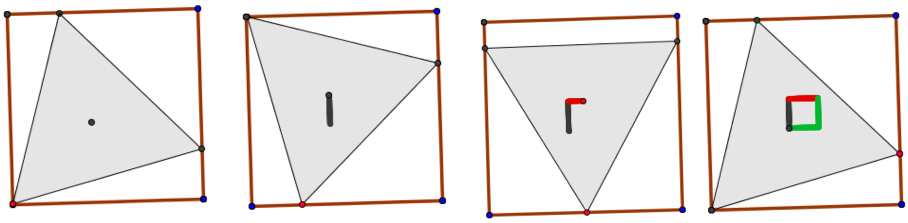

Fig. 8 The **(3;4)** dynamic construction

When we move the red point (**M**) until the next vertex of the triangle coincides with a vertex of the square (i.e. takes its initial position) we observe the trace becoming a shape which looks as a half of square. By analogy, when moving the point **M** along the rest of the sides of the square the center of the triangle will leave a trace which completes a square-like shape and after which it will start repeating the trace (three times). If the considered locus of the **(3;4)** construction is a square indeed could we conjecture that the corresponding locus of the **(3;5)** construction would be a regular pentagon? In the latter case it is sufficient to observe the effect of the movement of the red point on a part of the pentagon only.

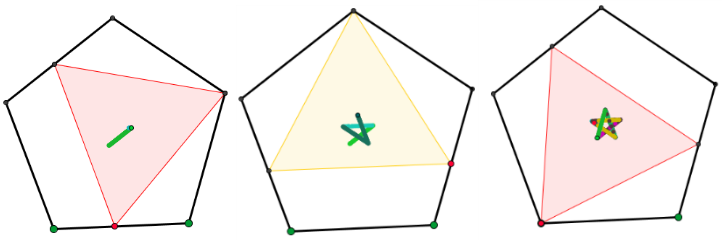

Fig. 9 The **(3;5)** dynamic construction

A-a-ah! Still 5 sides but it does not look like a pentagon – rather like a pentagram! Then what we suspected to be a square could be considered maybe as a „4-side star"...

Again, the center of the triangle describes the locus three times while the red point makes a full round along the original pentagon.

In the **(3;6)** case the locus appears to be a single point:

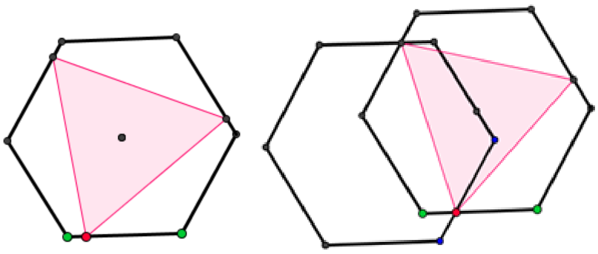

Fig. 10 The **(3;6)** dynamic construction

Such was the locus in the **(3;3)** case. By analogy we could conjecture that the same would hold for **(3;9),** and more general – for **(3;3*k*).** We could make separate construction for the **(*m*;*km*)**.

Further explorations providing insight

The (m; km) model

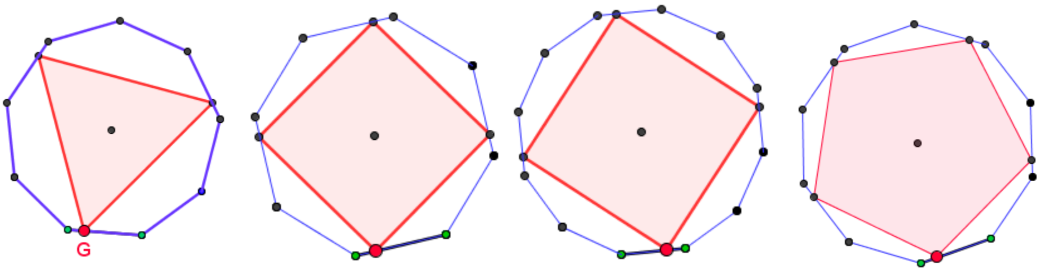

Fig. 11 The **(*m*;*km*)** dynamic constructions

The **(*m*;*km*)** constructions could be also achieved by analogy of the methods in Fig.6. The general conjecture we could draw after exploring the **(*m*;*km*)** model is that *for every point* ***G*** *on the* ***n****-gon (****n****=****km****) there exists an inscribed* ***m****-gon with a vertex* ***G*** *and the locus under consideration is a single point coinciding with the center of the* ***n****-gon.*

Let us continue our explorations with the (**3;*n***) model. In the case of **(3;7)** for instance we are expecting a star with its generating module emerging when going along one of the heptagon's sides. Indeed (Fig. 12)!

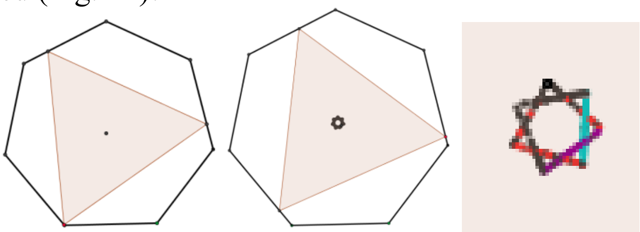

Fig. 12 The **(3;7)** dynamic model

Exploring further the (**3;*n***) model leads us to the conjecture that it is possible to inscribe an equilateral triangle in every regular ***n***-gon, i.e. (**3;*n***) *is always a possible construction*.

It is interesting to see what is the situation in the case of the (**4;*n***) model (Fig. 13).

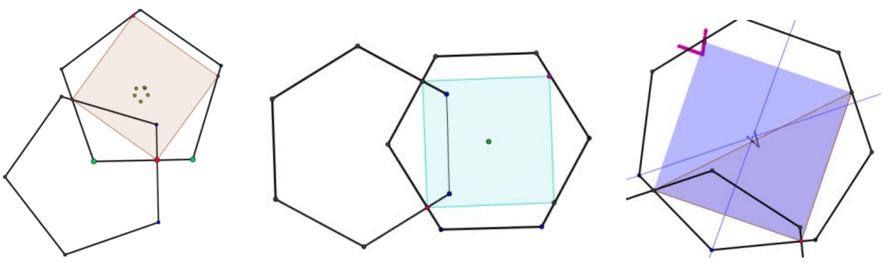

Fig. 13 The **(4;5), (4;6)** and **(4;7)** dynamic models

For a number of specific cases for $m > 4$, it is easy to make the conjecture that the construction is not always possible. In some cases additional means are needed for the inquiry. For example, in the **(5;6)** model it appears at first glance that the fifth vertex is on the hexagon (Fig.14 *a*). But a more careful exploration (Fig. 14 *b* and 14 *c*) shows that this is not so.

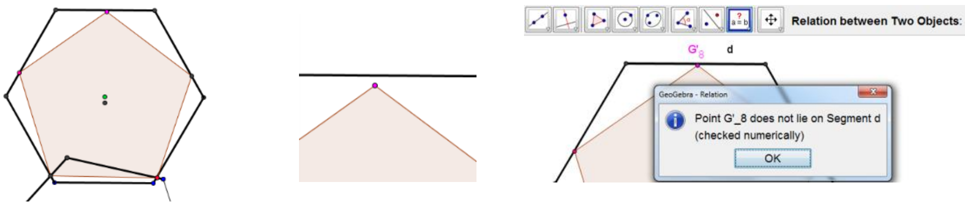

Fig. 14 The **(5;6)** dynamic model

At this point it is a good idea to stop and take a look around – *what is known in relation to our explorations?* We entered the magic phrase ***a regular m-gon inscribed in a regular n-gon*** and here it appeared (Dilworth, Mane, 2010)! Almost the same title and the same denotation showing how natural it is in its simplicity and conciseness when exploring various cases and describing the conjectures and results. Dilworth and Mane present there the necessary and sufficient conditions on ***m*** and ***n*** for inscribing a regular ***m***-gon in a regular ***n***-gon. It is interesting to note that *naively* (their own phrasing) they expected *this problem to be solved in the time of Euclid, but it seems to be not completely solved.*

Here is what Dilworth and Mane prove in (Dilworth and Mane, 2010) by means of complex numbers:

> **Theorem.** *Suppose that m, n* ≥ 3*. A regular m-gon can be inscribed in a regular n-gon if and only if one of the following mutually exclusive conditions is satisfied:*
>
> (a) $m = 3$;
> (b) $m = 4$;
> (c) $m \geq 5$ *and m divides n;*
> (d) $m \geq 6$ *is even and n is an odd multiple of* $m/2$. *(Note that this includes the case* $n = m/2$.*)*
> *In (c) and (d) the polygons are necessarily concentric and in (d) they share a common axis of symmetry. In case (d) we insist that n be an odd multiple of m/2 because if n is an even multiple of m/2, then n is a multiple of m, which is already covered in case (c).*

Thus it follows from the Theorem that the locus we are interested is a single point in the cases (c) and (d). The last examples of our explorations belong to (d).

Had we seen this article before attacking it with dynamic means we would feel very reluctant to offer it to students (even if they were very motivated to explore new mathematical territories). However, the explorations themselves harnessed mathematical skills accessible to students knowing about geometric transformations. Furthermore, the patterns and the relationships observed during these explorations gave rise to other interesting questions.

What really matters for us in relation to this problem is not even the solution itself but the whole process of creating a good platform for explorations, enhancing our intuition and understanding about some patterns among the constructions, designing a more systematic approach of explorations, realizing that not all combinations of inscribing a regular ***m***-gon in a regular ***n***-gon are possible, and finally – the belief in teachers' ability to promote the inquiry-based learning of mathematics. In a nut shell, to illustrate the „grook“ (*Grooks*, 2013) of the great Danish mathematician, architect and poet Piet Hein: *Problems worthy of attack, prove their worth by hitting back.*

ACKNOWLEDGEMENT

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STUDYING FINE-ART COMPOSITIONS BY MEANS OF DYNAMIC GEOMETRY CONSTRUCTIONS

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Abstract

The paper deals with integrating the study of art and mathematics by exploring the balance and the logical emphasis of paintings by means of dynamic geometry constructions developed in mathematics classes. A dynamic scenario developed and experimented in the context of the DynaMAT Comenius project is discussed as an illustration of how students could be encouraged to apply their mathematical knowledge for gaining a deeper insight in art compositions. Several methods suggested by art specialists are considered (e.g. rabatment, the rule of thirds, the golden section) together with appropriate dynamic geometry implementations. Ideas for further activities with students formulated in terms of long-term projects are offered. Although the considered scenario is still in its early phase of experimentation (mainly at teacher training courses) the first impressions are shared as being promising – the teachers become aware that they could attract more students to mathematics when showing its application in various contexts. The experience gained by the authors on a broader scale ─ in the context of visual modeling, is reported to have contributed to building new strategies in teacher education, which could prepare teachers for their changing role of partners in a creative process.

Keywords

dynamic geometry software, art, rabatment

INTRODUCTION

Seeing is not as simple as it looks
Ad Reinhardt

Many artists claim that they could explain nothing about their works, that their paintings came upon them by inspiration. The founder of the abstract art however, expresses in his book (Kandinsky, 2011) his theory of painting and sums up ideas that influenced his contemporaries. Kandinsky makes the brave prediction that *we are fast approaching the time of reasoned and conscious composition, when the painter will be proud to declare his work constructive*.

To motivate better the study of geometry for students with interests in art we could reveal for them the strong relation between the esthetics of artistic compositions and some geometric principles. When reading the works of art critics we come across notions such as *harmony, style, rhythm, balance* (not necessarily the better defined *rules, symmetry, geometry*). Perhaps they think that if revealed the rules behind a balance composition would trivialize the art. To us, revealing certain patterns and rules would in contrast raise the level of appreciation of an observer. The modern fine art tries to speak about things which *will be seen*, that is why its language is not understandable for many. But this language could be better learned if we try to study it together with the language of geometry.

VISUAL MODELING

Exploring the properties of geometric shapes in a computer environment has proven to be more exciting for students of different age if made part of a visual modeling of some works of art (Sendova and Girkovska, 2005, Nikolova et al, 2011). By building computer models of a given painting the students can gain deeper insight in its structure and motivation to elaborate their knowledge in mathematics and informatics.

When analyzing an abstract painting from mathematical point of view it is interesting to discuss its basic elements and to classify them. From an artistic point of view, though, the problem is not only to understand the elements of a composition, but also to understand its balance. In pre-service and in-service teacher training courses on using language-based computer environments for education we used a specially designed Logo microworld (Sendova, 2001) in which it was easy to experiment with figures of various sizes, colors and degrees of complexity, i.e. to verify different definitions of *balance*. In addition, the participants in the courses could play with Kandinsky's ideas concerning the relation between geometric shape and color and study the effect of both components in various combinations.

We could qualify the following factors as the most relevant ones in the study of an abstract painting:

- The character of the objects and their composition in terms of clustering, overlapping, isolation, balance, relationship between size, shape and color
- Main categories of the objects
- Establishing hierarchy related to the distance of the center, the size, the color, etc.
- Functional associations (which objects occur in combination in the work of a given author).

The visual modeling could be used not only to study a specific painting, or a specific artist, or more general – the style of a certain artistic movement, but also to bring possibly new creative ideas. Thus, products of the visual modeling should be judged with respect not only to the closeness between the original and the generated works but also to their potential to generate works bringing the spirit of the original together with new, unexpected ideas – a potential that depends on the user, of course. After leaving the frames of the strict imitation some of the future teachers were inspired by new combinations of forms and colors and got new insight, which in turn led to new formalization.

These visual modeling activities were carried out by means of programming which might create certain psychological problems among the typical mathematics teachers. Still combing art with geometry seems a very natural way of motivating the students to enhance their understanding in both fields. Some inspirational sources for integrating mathematics and art include (Ghyka, 1946, Livio, 2002, Hemneway, 2005, Olsedn, 2006, Skinner, 2009). More recent developments offer various dynamic geometry constructions as tools for analyzing works of art and appreciating the esthetics of well known paintings (Sánchez, 2013). The ideas presented below are based on a dynamic scenario (Sendova and Chehlarova, 2011) developed and experimented in the context of the *DynaMAT* Comenius project (http://www.dynamatadmin.oriw.eu) with the intention to encourage students in applying their mathematical knowledge for gaining a deeper insight in art compositions.

CREATING DYNAMIC CONSTRUCTIONS OF COMPOSITION TOOLS

The dynamic scenario deals with several relatively simple geometric constructions which have proved useful in creating and studying the balance of the fine-art compositions. After describing them we present their implementation in a dynamic software environment (*GeoGebra* in our case) so as to illustrate how they could be applied to exploring various paintings (classical and more modern alike). A further step offered to the students is to apply their newly gained art-evaluation competencies in the context of taking and editing photographs.

RABATMENT

The first (relatively less known) compositional method we introduce is *rabatment* which has been broadly used in the 19th century. This method is applicable to paintings in rectangular shape. It consists of taking the shorter side of a rectangle and placing it against the longer side (rotating the shorter side along the corner), creating points along the edge that can be connected directly across the canvas as well as a diagonal from these points to the corners. In a rectangle whose longer side is horizontal, there is one implied square for the left side and one for the right; for a rectangle with a vertical longer side, there are upper and lower squares. In traditions in which people read left to right, the attention is mainly focused inside the left-hand rabatment, or on the line it forms at the right-hand side of the image (Fig. 1).

Fig. 1 The left rabatment and its appearance in the Monet's painting *Red Poppy-field*

To achieve a more powerful composition one could add the diagonals of the rectangle and the two squares. Here is how the rabatment applied to the painting *A Sunday Reading in a Village School* of Bogdanov-Belsky looks like (Fig. 2).

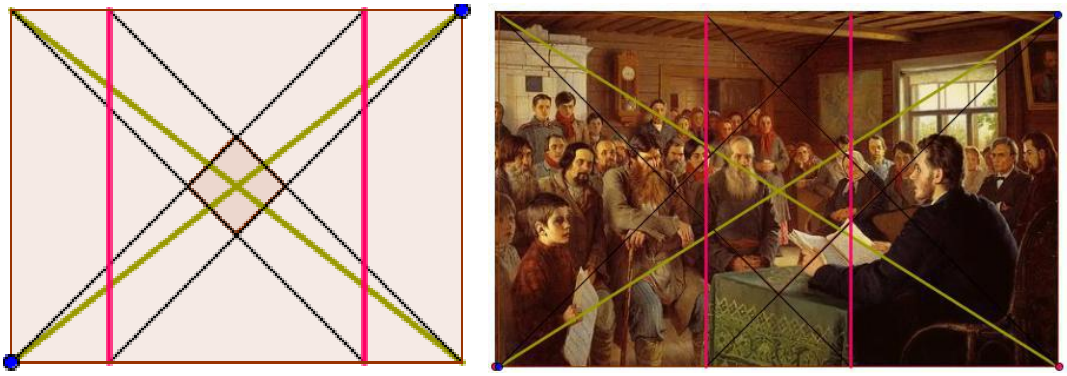

Fig. 2 The rabatment as applied to a Bogdanov-Belsky painting

More information about the rabatment method can be found at http://emptyeasel.com/2009/01/27/how-to-use-rabatment-in-your-compositions/ (Mize, 2013).

To enable the analysis of paintings of any rectangular shape it is convenient to have a universal rabatment tool. It could be created as a *dynamic geometrical construction* which means a construction which saves its main properties under movement of some of its objects defined as independent. The rabatment construction is introduced in our dynamic scenario by means of *GeoGebra* as follows.

CREATING A DYNAMIC RABATMENT TOOL

We start with constructing a dynamic rectangle. If the students are novices to using the software they should be encouraged to suggest and try out various ways of constructing a rectangle and discuss which of their constructions are in fact dynamic ones. After the discussion we could consider the following construction as appropriate for our purpose.

We construct one of the corners of the rectangle as an independent object – point *A*, and two variables (sliders) for its base *a* and height *b* specifying the range of their values. Next we construct point *B* - the opposite corner of the diagonal through *A* as a point whose coordinates depend on the coordinates of *A*, *a* and *b*). Then we construct lines through *A* and *B* parallel to the coordinate axes. The remaining two vertices of the rectangle could be obtained as intersection points of these lines. Then we connect the four points to get a rectangle (which is dynamic with respect to the size of its sides but always with a horizontal base, the normal position of a painting's frame).

Let $b < a$. Now we construct circles with centers the four vertices of the rectangle and a radius *b* – the length of the shorter side of the rectangle. We find the intersection points of the four circles with a side of the rectangle and construct two of the rabatment segments. Then we complete the construction with the diagonals.

What is left is to construct the square in the center which appears when $b < a < 2b$. We hide the auxiliary objects (the lines and the circles) and we explore the construction for various values of *a* and *b*.

Similarly, we make a workable construction for a rectangle with a shorter base.

It is worth mentioning here that a good educational quality of *GeoGebra* is the opportunity for the users to enrich the toolkit with their own tools. This facilitates the implementation of our rabatment tool, viz. we show to the students how to make the rabatment construction a part of the toolbar. For the purpose a suitable name, an icon and the inputs of the construction (a point and two numbers in our case) should be specified.

In our scenario we have considered in fact two constructions appropriate for a rabatment tool, the second one having as inputs the ends of one of the rectangles diagonals. Thus the students could create and use two rabatment buttons in the same *GeoGebra* file (named RabatmanPNN and RabatmanPP after the necessary inputs for the respective construction). As a further step in our scenario we show how images could be studied by means of the composition tools being created, i.e. how to display and how to resize (if necessary) the inserted image by preserving its proportions. Now the ground for explorations is set – the students could use the rabatment button, place the rabatment construction on the image and look for interesting properties of the composition of a specific painting (Fig.3).

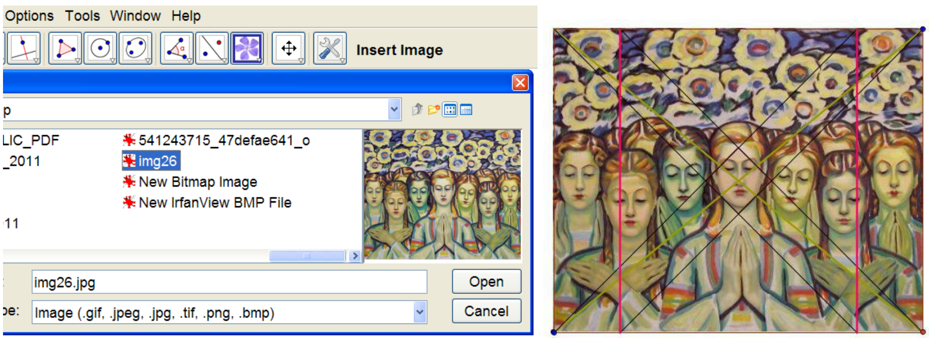

Fig. 3 Inserting the painting *Prayer* by Maystora and applying the Rabatment button to it

Depending on the emphasis the teacher would like to make, s/he could encourage the students to continue with exploring the created dynamic tool with other paintings and to formulate their findings. Alternatively s/he could enhance their mathematics skills of implementing other composition tools. Here is what we have suggested further in our scenario.

THE RULE OF THIRDS

The *rule of thirds* is a simple method that can be used not only as a tool for exploring the paintings of famous artists but also to enhance and improve our own compositions (when we draw or take pictures). In the diagram below, a rectangle has been divided horizontally and vertically by four lines. The rule of thirds states that the points of interest for any rectangle are determined by those lines. The intersections of the lines are considered by some specialists (Maze, 2013) to be *power points* (the black dots in Fig. 4).

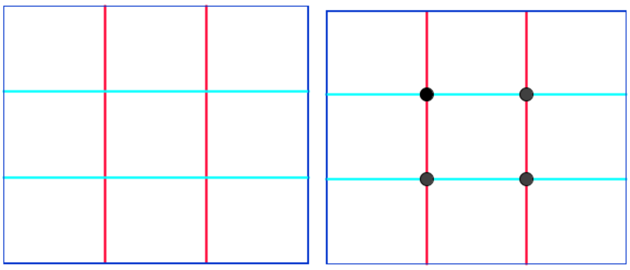

Fig. 4 The rule of thirds and the *power points*

Here is the rule of thirds in action (in horisontal version and in vertical one):

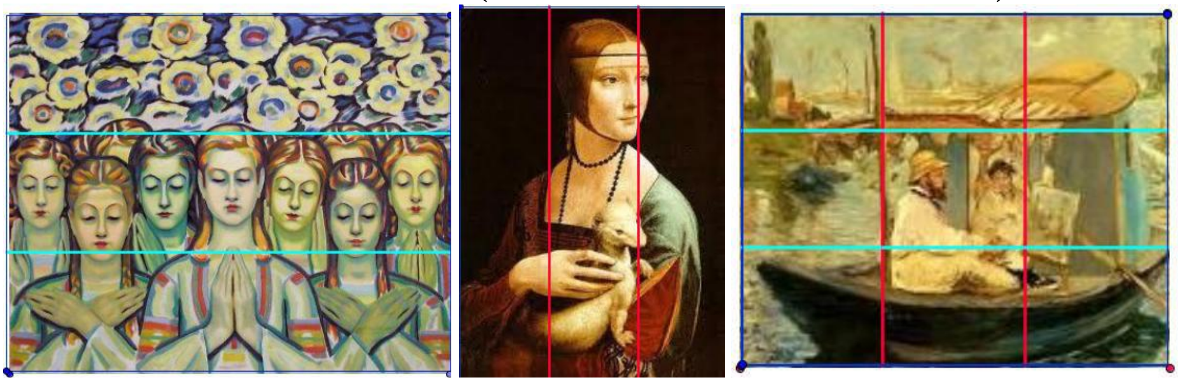

Fig. 5 The *rule of thirds* applied to Maystora's *Prayer,* to Leonardo's *Lady with an Ermine,* and to Manet's *Monet painting in his floating studio*

For the students it is of essential importance to use the rule of thirds not only when studying famous paintings but also when taking (or editing) photographs of a scenery (Fig. 6).

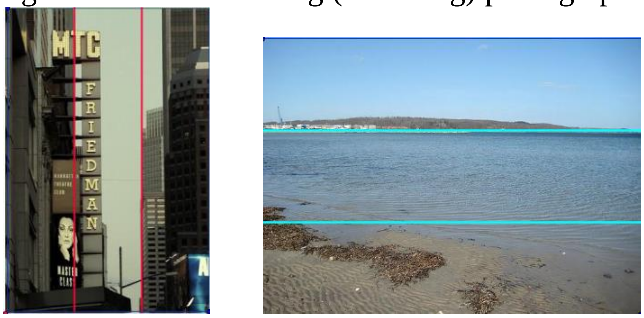

Fig. 6 The *rule of thirds* in photography

The teachers could create a list of tasks for the students taking into account their interests and mathematics background. Some examples of tasks we have offered in our dynamic scenario deal with

- Making several digital pictures of a scenery by applying the rule of thirds in just one of them and explain which version seems to be the most balanced one
- Creating *Thirds* buttons (a vertical and a horizontal versions)
- Exploring some classical and some modern paintings by various composition buttons.

In addition to using some classical composition tools the mathematics teachers could suggest geometric constructions of their own, possibly jointly with the art teachers. Here is an example from our scenario.

THE CENTRAL RHOMBUS

The logical emphasis of a painting is often located in a rhombus with vertices the midpoints of the sides of the rectangle:

Fig. 7 The central rhombus applied to Maystora's *The Girl with the Dahlias*, and to Mrkvička's *Ruchenitsa*.

To make a dynamic construction and turn it into a rhombus button in *GeoGebra* for exploring images is another activity offered to the students.

Of course, the most popular notion combining art and mathematics is the *golden section*.

A DYNAMIC GOLDEN SECTION CONSTURCTION

The most famous mathematical composition tool, though, is the *Golden Section* (also known as the *Golden Mean* or the *Golden Ratio*) defined as the point at which a segment can be divided in two parts *a* and *b*, so that $a/a+b = b/a$. We introduce the notion of a *golden rectangle* as a rectangle whose side lengths are in the golden ratio. The golden ratio is often depicted as a single large rectangle formed by a square and another rectangle. What is unique about this is that we can repeat the sequence infinitely within each section. If in addition we draw an arc of 90^o in the consecutive squares we get the so called *golden spiral* (Fig. 8).

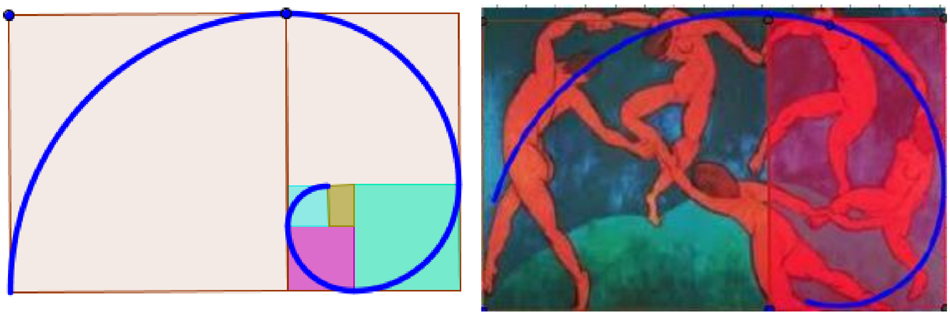

Fig. 8 A sequence of golden rectangles and the golden spiral applied to *The Dance* by Matisse

We give in our scenario the following algorithm for constructing a dynamic golden spiral

- *Construct a unit square (blue).*
- *Draw a segment from the midpoint of one side to an opposite corner.*
- *Use that segment as the radius of an arc that defines the longer dimension of the rectangle (Fig. 9).*
- *Construct an arc of* 90° *in each square so as to get a golden spiral:*

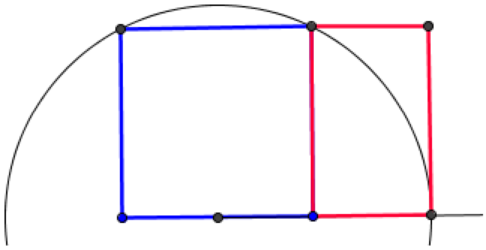

Fig. 9 Constructing a dynamic golden rectangle

Then we suggest to students to construct *GeoGebra* buttons based on the golden ratio and to explore various paintings with all the composition tools they have created.

It is very important to extend these activities by assigning long-term projects to the students. Here are some examples of *dynamic mini-projects* included in the scenario:

- *Take a picture of a scenery in two ways so that they reflect specific goals. Explore the result by means of dynamic constructions and edit the pictures correspondingly by cutting out.*
- *Arrange for a picture in two ways (according to two composition methods): 6 persons at a birthday party sitting around a round table; a class of 24 pupils and their teacher; flowers and fruits; perfumes and an advertisement. Explore the result with dynamic constructions and make corrections if necessary.*
- *Make an advertisement in two ways of: your school; your hobby; natural juices; an old town. Explore the result with dynamic constructions and make corrections if necessary.*
- *Make in two ways a design of an invitation cart for: a fest of mathematics (physics, music, the flowers, athletics); a ball with masques; a birthday party. Explore the result with dynamic constructions and make corrections if necessary.*
- *Explore the rotational dynamic constructions by means of the sliders so as to create models similar to the pictures of rotational objects (Fig. 10).*

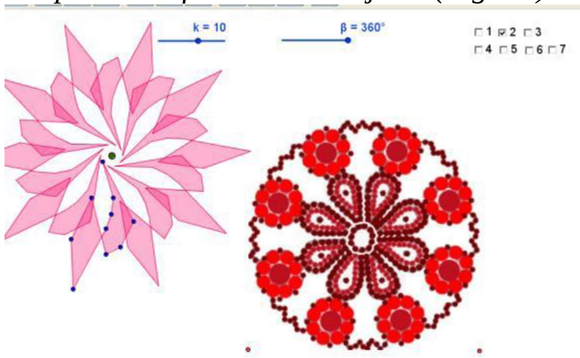

Fig. 10 Rotational dynamic construction

- *Create models of objects around you based on rotational symmetry (wood carved ceilings, embroidered table clothes, etc.)*

CONCLUSIONS

Our overall experience in educating students and teachers alike shows that the integration of the learning and creative processes by means of visual modeling could contribute to a new learning style in mathematics education. Such type of activities sensitizes students to looking at not only the art but also at the world around them in a more meaningful way.

Although the considered scenario is still in its early phase of experimentation (mainly at teacher training courses) the first impressions are promising – the teachers become aware that they could attract more students to mathematics when showing its application in various contexts (often unexpected for them as art is).

A famous quote by the american poet Robert Frost reads: *Writing free verse is like playing tennis with the net down.* We could extend this quote to art in general. But an important point we make to the teachers is that every rule can and should be broken for artistic effect, from time to time. This should be done however not because we don’t know the rules but rather when we are looking for new ideas. This is for example how some stunning photographs are made.

The experience gained leads us naturally to building new strategies in teacher education, which could prepare teachers for their changing role of partners in a creative process.

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FIELDWORK AS A TEACHING METHOD - A CASE STUDY USING GPS

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Abstract

In this paper we want to demonstrate a case study showing how fieldwork can be used as a teaching method. We will provide a definition of this teaching method, explain its advantages and disadvantages, describe the background and the setting of the case study, and report about the evaluation results. Fieldwork is rarely used in several school systems, both for legal and practical reasons. Yet it can increase students' motivation, particularly in mathematics, as students can experience first-hand "what can mathematics be used for?" We will present a teaching unit using GPS as a tool to construct or measure geometric figures in the field, i.e. outside the classroom. This gives students an opportunity to learn about geometric figures not only in an abstract setting, but as shapes used in the real world. The questionnaires and interviews that we conducted show that this teaching unit improved students' motivation to find out more about real-life uses of mathematics, as well as the possibility of increasing students' attitudes towards learning mathematics by providing possible applications.

Keywords

Fieldwork. Real-life tasks. Electronic media as tools for learning

DEFINITION AND INTRODUCTION

By fieldwork we mean work of students outside the classroom. It may involve work in the school grounds or further afield. It can vary in duration – part of a lesson, a half day, or longer. It involves live collection of primary data by means of observation, experiment or survey (Ulovec et al., 2007). In this way, students can experience familiar and unfamiliar phenomena beyond the normal confines of the classroom (Dillon et al., 2005).

However, fieldwork is not frequently used in many classrooms. This might be because of practical reasons, because of legal hassles, or simply because of a lack of teaching materials with proper suggestions. As for the practical reasons, some tips and hints can be found in Simperler (2012). As for the legal hassles, it might be true that using fieldwork requires some form-filling, parents to be contacted etc. But this is also true for other out-of-school activities, e.g. ski courses, swimming weeks, excursions etc., and so should not prevent one from using this method. As for the lack of teaching materials, this is the main reason for writing this paper. We developed a number of out-of-classroom activities in an EU-funded project called DynaMAT. One of the materials is presented here, together with an evaluation in the form of questionnaires and interviews, to serve as a case study about the usefulness of this teaching method.

ADVANTAGES AND DISADVANTAGES OF FIELDWORK

Several authors have already dealt with this issue, and a number of advantages and disadvantages have been listed in the literature. The following table presents a summary of this work (cf. Sauerborn and Brühne, 2009):

Tab. 1 Advantages and disadvantages of fieldwork as a teaching method

Advantages	Disadvantages
Action-oriented	Difficult with large number of students
Reality-related	Organisational effort
Physical activities	Risks of injury
Self-responsible learning	Difficult assessment

New method for most students	Students not used to this activity
Addresses several cognitive learning levels	Hard for students to concentrate
Often interdisciplinary	Hard to place in curriculum

In a 1999-study of the University of Regensburg about out-of-classroom learning activities, the most frequently named disadvantage was “costs” (53.3%), closely followed by “time pressure by curriculum” (51.7%). In our own study (see below), costs were not an issue to teachers, as most activities took place either on the school grounds or in walking distance of the school. Time pressure by curriculum was the most frequently named reason (65.7%), followed by fear of disciplinary issues (37.1%), and organisational effort (28.6%).

CASE STUDY: USING GPS IN FIELDWORK

Teaching material

This teaching unit (cf. Andersen, 2012) consists of two parts: In the first part, students are asked to use the tracking function of a GPS receiver to measure the geometric shape of a given outdoor feature. In the second part, the students are given a certain geometric figure and are asked to “walk along” this figure outdoors, i.e. to use the GPS receiver to navigate in such a way as to produce a track in the form of the given geometric figure.

Part 1: Measuring a geometric figure in the field

Task: Go to Heldenplatz in Vienna (or a park nearby the school) and stand on one corner of the rectangle that is shown in the map below (or another suitable rectangular figure in the park). Switch on the tracking function of your GPS receiver. Now walk along the edges of the rectangle until you are back at the original point. Then switch off the GPS receiver. Compare the resulting track with the original rectangle. Use the obtained data to calculate the side lengths of the rectangle and the length of the diagonal. Then go back to Heldenplatz (or the chosen park) and measure the length of the diagonal with the GPS receiver.

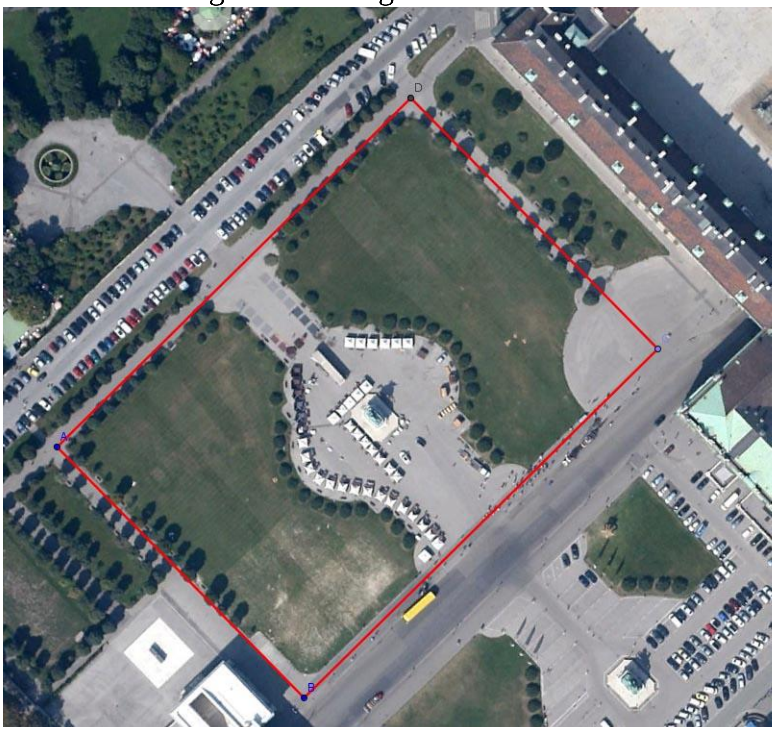

Fig. 1 Measurement of rectangle – ideal track

Part 2: Creating a geometric figure in the field

Go to Heldenplatz (or a suitable nearby park or field) again and use your GPS receiver to walk along an equilateral triangle with a base length of 40 m. Before you set off, think about how to do this, what strategies are possible, what their advantages and disadvantages are, and which one you will choose. Switch on the tracking function of the GPS receiver and record your “triangular walk”. Transfer the data into Google Earth and check with GeoGebra how close your track comes to an exact equilateral triangle. Compare your results with those of your classmates, particularly with those who have chosen another strategy than yourself.

Setting

The teaching unit was performed in 13 secondary school classes in 8 schools in Vienna, with a total number of 223 students and 35 teachers involved. The teachers received the teaching materials and – if required – a number of GPS receivers. The students received an instruction into GPS as such (using Ulovec, 2012a) and an instruction on how to transfer and interpret GPS data with Excel (using Ulovec, 2012b), Google Earth and GeoGebra (using Andersen, 2012). These instructions took two lessons (50 minutes each) per class. The fieldwork as such was led by the mathematics teacher with the support of 1 – 2 colleagues (also teachers, but mostly of other subjects). Part 1 took one lesson, part 2 took two lessons of 50 minutes each. Part 2 was usually (with 2 exceptions) done in a double lesson of 100 minutes in one piece. After the teaching units were conducted, the teachers and students were given questionnaires about the concrete teaching units and the teaching method “fieldwork”. The students’ questionnaires did not contain mathematical tasks (i.e. it was not a pre-post-test setting), but did make some references to the geometrical content. 5 teachers and 22 students were also interviewed after the teaching units.

Description

As to part 1, students were usually able to walk the path as described and record the data. A typical track looked like this:

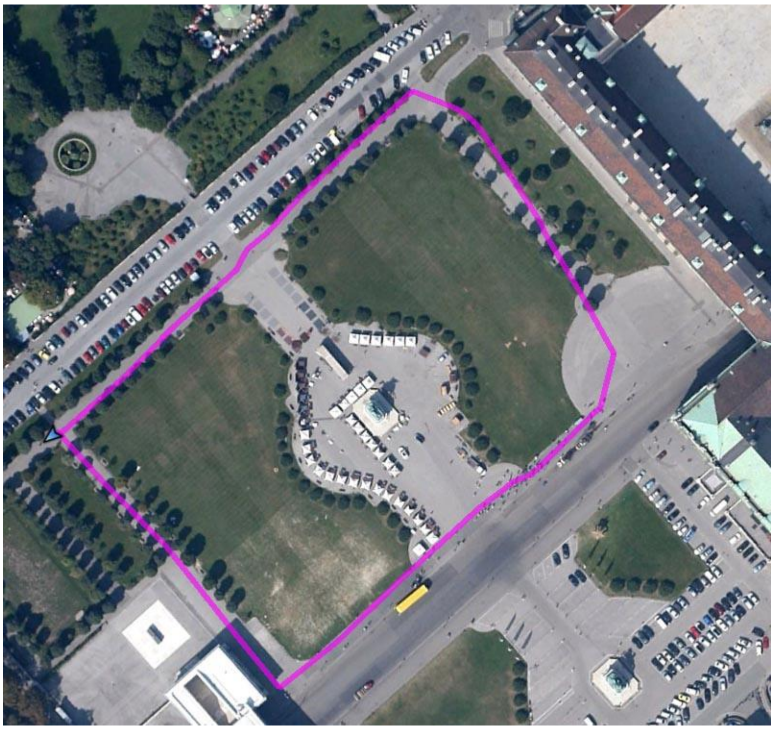

Fig. 2 Measurement of rectangle – real-life GPS track

Measurements of the diagonal length resulted in fairly exact data, 210 (of 223) students were within a 15% error margin.

In part 2, we could observe mainly two strategies: Using angles, and using vertices as waypoints:

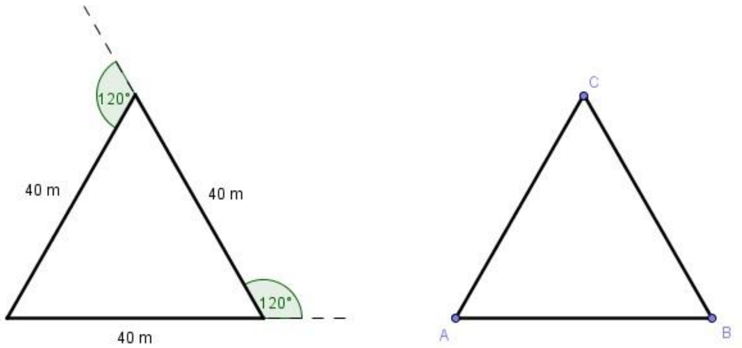

Fig. 3 Angle strategy waypoint strategy

Strategy 1 calls for the student to walk 40 m, then turn 120° anti-clockwise, walk another 40 m, turn again 120° anti-clockwise, and walk another 40 m. Strategy 2 calls for the student to calculate (or construct in GeoGebra) the coordinates of the vertices and set them as waypoints in the GPS receiver before starting out, then walk to the first waypoint, the second waypoint, the third waypoint, and then back to the first waypoint.

Typical results of strategies 1 and 2 look like this:

Fig. 4 Triangle – strategies 1 and 2

Teachers and students then discussed the advantages and disadvantages of the two main strategies that occurred. Main points that were mentioned were: Strategy 1 has the advantage of not requiring any pre-fieldwork calculations (except for figuring out that the outer angle of an equilateral triangle is 120°). It has the disadvantage of not being too accurate in the field. Strategy 2 requires some calculations and constructions with GeoGebra or similar tools, and additional operations with the GPS receiver. However, it leads to better results.

Questionnaire and interviews

After finishing the activities of the teaching units, both the teachers and the students received (different) questionnaires. Aside from personal data (grade for students, teaching experience for teachers etc.) there were the following quantitative questions for students, to be answered on a scale from 4 (very much) to 1 (not at all):

1s) Was the teaching material adequate for the lessons?
2s) Did you know/learn all the technologies that you needed for these lessons?
3s) Did these lessons change your attitude towards mathematics in a positive way?
4s) Did these lessons increase your motivation to find out more about practical uses of mathematics?

Teachers received the following quantitative questions:

1t) Was the teaching material sufficient for the lessons and their preparation?
2t) Did you know/learn all the technologies that you needed for the preparation and execution of the lessons?
3t) Did these lessons change your attitude towards fieldwork as a teaching method in a positive way?

Both groups received the following qualitative questions:

5) What did you like the most about the teaching material?
6) What did you like the least about the teaching material?
7) What did you like the most about the teaching method “fieldwork”?
8) What did you like the least about the teaching method “fieldwork”?

Teachers were additionally asked:

9t) What do you see as the three biggest advantages of the teaching method “fieldwork”?
10t) What do you see as the three biggest disadvantages of the teaching method “fieldwork”?

After the analysis of the questionnaires, we chose 5 teachers and 22 students, and interviewed them about their answers to some of the quantitative questions, and all qualitative questions. The interviews lasted about 20 minutes per interviewee.

Results

207 students and 35 teachers handed in the questionnaires. Here are the results of the quantitative questions. Given is the percentage of answers on a scale of “4” (very much) to “1” (not at all).

Tab. 2 Results of questionnaires, questions 1 – 4

	Teachers (n = 35)				**Students (n = 207)**			
	4	3	2	1	4	3	2	1
Question 1	80	17	3	0	84	9	5	2
Question 2	74	20	6	0	77	10	8	5
Question 3	49	28	17	6	38	18	28	16
Question 4					30	39	16	15

As to the teaching materials and the technologies used, we can clearly see that they were very well accepted. Also the interviewees confirmed this observation. The only issue for students was the inaccuracy of the GPS receiver, which particularly occurred when students did not use a stand-alone receiver but their smartphone or similar device.

As to the attitude aspects, there was no significant improvement by the teaching units alone. However, in the interviews those students who claimed that their attitude towards mathematics has changed in a positive way, almost unanimously stated that this is because the teaching unit showed “what mathematics can be used for, except in school” or “real-life applications”.

The teaching units definitely increased students’ motivation to find out more about practical uses of mathematics (69% of students answered either with 4 or 3 to this question).

27 out of the 35 teachers answered with 4 or 3 to whether the lessons changed their attitude towards fieldwork in a positive way. This was also confirmed in the interviews, where teachers (most of whom have never used fieldwork as a teaching method in a regular lesson) stated that this was a good opportunity for students to use mathematics outside the classroom, and experience geometric figures that are not just drawn in their notebooks or displayed on a computer screen. Also, in the interviews many teachers stated that fieldwork either requires very well prepared teaching unit descriptions, as delivered here, or a lot of effort from the teacher to develop and prepare suitable units themselves.

In the qualitative questions with respect to the teaching materials, in question 5 both teachers and students commented positively on the use of GPS technology, which is not usual in mathematics teaching, and seemed to be very motivating for students (as was mentioned in several interviews). Also "practical example" and "good instructions" have been mentioned frequently, both by teachers and students. In question 6, "too technology-centred" and "hard to fit into curriculum" was mentioned by teachers, "better use rectangle instead of triangle, for comparison" and "would have been better if everyone would have their own GPS unit" was mentioned.

In the qualitative fieldwork questions, at question 7 students mostly answered "to work outside" and "it is not boring", teachers answered "seems to be motivating for students" and "allows the teachers to show application in real life instead of just explaining it in classroom". At question 8, only few students and teachers gave any answers, mostly along the lines of "a lot of work for a maths class".

In the final two questions for teachers, the three most frequently named advantages were "opportunity to show real-life applications" (51.4%), "motivation for students" (37.1%), and "physical exercise" (22.8%). The three most frequently named disadvantages were "time pressure by curriculum" (65.7%), "fear of disciplinary issues" (37.1%), and "organisational effort" (28.6%).

CONCLUSION

Fieldwork is a teaching method that can help students to see possible applications of mathematics in real life, outside their classrooms, and by that increase their motivation to look for more applications of mathematics in their lives. It is clear that it requires appropriate preparation, both with respect to the actual teaching unit, and with respect to organisation. However, as this case study shows, and other authors confirm (e.g. Scherer and Rasfeld, 2010), it is a good opportunity for students to widen their views of mathematics and prevent it to become a classroom-only activity.

In any case, most of the literature in this field concerns work with very young children (e.g. Dühlmeier, 2008; Stevens and Scott, 2002), and there is not all too much about fieldwork in mathematics with secondary school students, so more work needs to be done for this particular age group.

ACKNOWLEDGEMENT

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PROBLEM POSING IN MATHEMATICAL EDUCATION: DIOPHANTINE EQUATIONS AND A PROBLEM IN GEOGEBRA

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Abstract
This paper is part of Project Comenius DynaMAT, so it shares with it motivation and scope: to create problems aimed at high school students in order to introduce them to mathematical arguments and reasoning, and to, via problem-solving, advanced topics in an elementary way; all of this is done using computer software to help visualisation and intuition. In this paper we particularly concentrate on the problem-conjecture-proof-problem process of mathematical reasoning, and on the limits showed by ICT, which helps intuition, but has to be supported by rigorous demonstration. In particular we will deal with two problems, one arithmetical problem based on Diophantine equations, and one geometrical problem analysed with GeoGebra. Solutions are given, but the real protagonists of the paper are the many possibilities of deeper study and exploration offered to the students along the solution and proposed generalisation of the problems. No conclusions are given, but offers of further evolution, the basis of which consists in actual experimentation of the materials produced in classrooms, followed by data collection and analysis, in a collaboration between researchers and teachers.

Keywords
Dynamical Approach, Divulgation And Didactics, High School Problems, Diophantine Equation, Geogebra, Excel

INTRODUCTION

In this paper we will analyse two problems prepared for high school students: we will describe solutions to these problems which can be presented to such an audience. Our main interest is to propose approaches that can, in our opinion, help the students understanding and interiorising the processes and ideas typical of mathematical reasoning. Moreover, we will introduce some topics and problems connected with Diophantine equations and modular arithmetics following problem-solving approach.

In doing so, computer programs like GeoGebra and Excel can be a great help, and we will describe their use. Moreover, we will also describe some risks that computers programs can hide, following the idea that while a computer graphic program can be a wonderful source of intuitions in the hands of an expert mathematician, it can also be a hint to wrong conclusions for a young student: we will therefore underline the importance of the formal demonstration part in mathematics, that has to follow necessarily the intuition.

More motivations and information about this approach can be found in (Georgiev, 2012).

ARITHMETIC PROBLEM: PRESENTATION AND SOLUTION

The problem is well introduced in form of real life stories or games in (Dimitrova et al., 2008), (Georgiev, Kurokawa, 2012-1), (Georgiev, Kurokawa, 2012-2) and (Anderson et al., 2010).

We briefly recall the game formulation: consider the equation $ax = by + c$, where a, b, c are natural number. x, y can be interpreted as two buttons (green and red) : if one presses the green button this corresponds to the operation $x = x + 1$, if one presses the red button, then $y = y + 1$ is performed. At first $x = 0$ and $y = 0$, who obtains the equality pressing the buttons as few times as possible wins.

After having played long enough, the students can be lead to a formal exposition of the problem, and its solution, which is what we are interested in, in this paper. We are particularly interested in the possibilities to introduce mathematical concepts and methods given by this elementary solution.

We start by suggesting one possible translation of the problem in mathematical language:

Problem. Given $a, b, c \in \boldsymbol{N}$ (set of natural numbers), consider the following Diophantine equation:

$$ax - by = c \qquad (1)$$

Find, if it exists, among the solutions of equation (1) the one, denoted by (x_0, y_0), such that $x_0 + y_0$ is minimum.

In other words, find (x_0, y_0) solution of equation (1) such that (x_1, y_1) solution of (1) implies:

$$x_0 + y_0 \leq x_1 + y_1.$$

Please notice that we just consider non-negative numbers.

First of all it is important to find out whether a solution always exists or not.
It is useful to this purpose to recall the concept of greatest common divisor, or simply gcd, of two integer numbers.
If we consider $m = \gcd(a, b) = (a, b)$, then it is clear that:

$$m \mid ax - by \qquad \forall x, y \in N;$$

So if m is not a divisor of c, there is no solution at all to equation (1).

It is an elementary fact that if m divides c, then equation (1) has solutions, and moreover all solutions are given by a closed formula, check (Herstein, 1972) for some details.
Anyway, our goal is to introduce young students to these topics, using this problem as an excuse, so let us show a possible solution which covers almost every aspect of the basic arithmetics.
We can recall the concept of Euclidean division between two integers, and the so called extended Euclidean algorithm to find the gcd and a linear combination of the two integer numbers that gives the gcd itself, namely two integers k, h such that:

$$ak + bh = (a, b).$$

An interesting way of explaining the Euclidean algorithm is using a spreadsheet application such as Excel.
A quick explanation of how the algorithm works can be found at (Wikipedia, *Extended Euclidean Algorithm*), while a good description of the implementation is given by (Mounth Olyoke College, *The Euclidean Algorithm in Excel*).
Anyway a quick recall is given in Fig1.

	A	B	C	D
1		a	b	
2				N = ax+by
3				
4		N	x	y
5		=B2	1	0
6	=INT(B5/B6)	=C2	0	1
7	=INT(B6/B7)	=B5-A6*B6	=C5-A6*C6	=D5-A6*D6
8	=INT(B7/B8)	=B6-A7*B7	=C6-A7*C7	=D6-A7*D7
9	=INT(B8/B9)	=B7-A8*B8	=C7-A8*C8	=D7-A8*D8
10	=INT(B9/B10)	=B8-A9*B9	=C8-A9*C9	=D8-A9*D9
11	=INT(B10/B11)	=B9-A10*B10	=C9-A10*C10	=D9-A10*D10
12	=INT(B11/B12)	=B10-A11*B11	=C10-A11*C11	=D10-A11*D11
13	=INT(B12/B13)	=B11-A12*B12	=C11-A12*C12	=D11-A12*D12
14		=B12-A13*B13	=C12-A13*C13	=D12-A13*D13

	A	B	C	D
1		a	b	
2		5915	2317	N = ax+by
3				
4		N	x	y
5		5915	1	0
6	2	2317	0	1
7	1	1281	1	-2
8	1	1036	-1	3
9	4	245	2	-5
10	4	56	-9	23
11	2	21	38	-97
12	1	14	-85	217
13	2	7	123	-314
14		0	-331	845

Fig. 1 Euclidean algorithm in Excel: formulae and an example

A very good way to help the students become acquaintance with both the algorithm and the potential of a spreadsheet application is to explain the algorithm and the basic functions of the

software, and then ask them to find an implementation of the algorithm: they have a quick way to check its correctness: just trying with some couples of numbers, they learn how to use a spreadsheet "naturally", i.e. actually using it to solve some problem they should care about. Moreover they discover the usefulness of abstract reasoning and formulae expression, over the simple analysis of some special cases.
Going back to the problem, thanks to the Euclidean algorithm it is now easy to obtain any multiple of (a, b) with combinations of a and b. Consider $d = n \cdot (a, b)$, then:

$$a(nk) + b(nh) = (a, b)n = \text{d}.$$

We can therefore simplify the problem, assuming $(a, b) = 1$, since if (a, b) does not divide c, then the problem has no solution, if (a, b)divides c, then call:
$m = (a, b), a = m \cdot a_1, b = m \cdot b_1, c = m \cdot c_1$, so that:

$$ax - by = c \leftrightarrow m(a_1 x + b_1 y) = mc_1 \leftrightarrow a_1 x + b_1 y = c_1$$

Thanks to these preliminary considerations, and especially to the programming of the Euclidean algorithm, a student should become acquaintance with the tools involved in this formalization of the problem.
Therefore, we can now go on proposing a solution.
Theorem 1. Given $a, b, c \in \boldsymbol{N}$ (set of natural numbers), the following Diophantine equation:

$$ax - by = c \qquad (1)$$

has one and one only solution (x_0, y_0) such that y_0 is an element of the set $\{0, \dots, a-1\}$ and x_0 is a positive number.
Proof. Since $(a, b) = 1$, also$(a, -b) = 1$, and we already proved via Euclidean Algorithm that exist k, h integer numbers such that:

$$ak - bh = 1$$

In general $k, h \in Z$, but considering the following auxiliary equation:

$$ax - by = 0,$$

which has solutions: $x = bt, y = -at$ for any $t \in Z$, it is easy to notice that any couple $(k + bt, h - at)$ is a solution to equation (1).

Finally we can consider the Euclidean division $h = qa + r$, with $0 \le r < a$, and find out that the couple:

$$(k + bq, h - aq) = (k + bq, r)$$

Is our solution, since r is in the set considered and is unique.
We just have to check x_0 is non-negative, and this is a simple computation:

$$x_0 = \frac{c + by_0}{a} \ge 0.$$

This concludes the proof.

Notice that what we did here is a simplification of the construction of general solutions to a Diophantine linear equation, so such topic could easily follow.

Notice also that the proof could be easily expressed in modular arithmetic terms, and this could be a fine way to introduce such formalism among young students too. In fact, we could have exposed the proof in this way:
Consider the modular equation:

$$-bY \equiv c \quad mod(a)$$

Since$(a, -b) = 1$, the Euclidean algorithm tells us that exists k such that:

$$ah - bk = 1.$$

This means by definition: $-bk \equiv 1 \ mod(a)$, we could write $k = b^{-1} \ mod(a)$.

So it is easy to compute $Y \equiv -cb^{-1} \quad mod(a)$.
Then it is obvious by definition that exists one and one only $y \in [0, a-1]$ such that $[y]_a = Y$.

To solve our problem we have to show that (x_0, y_0), the solution to equation (1) given by the last theorem is also a solution to our original problem.

Theorem 2. In the same hypothesis of the previous theorem, given $(x_1, y_1) \in N^2$solution to equation (1), different from (x_0, y_0) then:

$$y_1 > y_0 \; and \; x_1 > x_0$$

Proof. The first observation is that (x_1, y_1) different from (x_0, y_0) implies both $x_1 \neq x_0$ and $y_1 \neq y_0$, quite obviously. Then since $y_1 \in \boldsymbol{N}$ and y_0 is the only possible y in $\{0, \ldots, a-1\}$ part of a solution to equation (1), we deduce $y_1 > y_0$.

It follows immediately:

$$x_1 = \frac{c+by_1}{a} > \frac{c+by_0}{a} = x_0.$$

Our proof is now complete, and the problem is solved.

DESCRIPTION OF THE FINAL ALGORITHM

Given equation:

$$ax - by = c \qquad (1)$$

We can find (x_0, y_0) solution to equation (1), such that $x_0 + y_0$ is minimum following these steps:

1) If (a, b)does not divide c, then the problem has no solution.
   If it does:
2) Compute, via Euclidean algorithm k, h such that $ak - bh = (a, b)$. Then consider the Euclidean division $h = qa + r$. Put $y_0 = r$.
3) Compute $x_0 = (c + by\_0)/a$.
4) (x_0, y_0) is the only solution to our problem.

Now that we solved this problem, it is possible to propose to the students some modification or generalization of it. The aim is to involve the students in the typically mathematical circle of problem-conjecture-proof-problem: once we have concluded a demonstration of our conjectures, we are pushed to analyze it and wonder: "What did we really show?" "What results similar to this could I face now?" "How could I generalize my results?"

Some of the more immediate and interesting generalization are:

Change of sign.

What if we considered the same problem, but with the sign "+"?

It is now clear that in some simple examples there is no solution (among positive numbers):

$$2x + 3y = 1,$$

even if $(2,3) = 1$. So we are facing a different kind of problem, and a first natural question is: will the approach we adopted in the preceding problem still work? If not, where does it fail? And how could we solve this new problem?

Problem with three variables.

What if we considered an equation with three variables instead of two, i.e.:

$$ax = by + cz + d,$$

and we tried to find a solution made of natural numbers, such that $x + y + z$ is minimum?

This particular example is interesting for the following reason: while it is "easy" to occur in the solution in the case of two variables linear equation, since in some sense it is the first

solution one finds (remember we proved y_0, part of the solution, is the only y in $\{0, \ldots, a-1\}$ part of a solution) if tries with $x = 0$, then $x = 1$ and so on, the first solution one finds is the good one. This does not always happen in the three variables linear equation problem.
Consider the following equation:

$$29x = 4y + 53z + 5$$

If one looks for a solution with $x = 1$, finds the solution $(1{,}6{,}0)$, which has sum $S_1 = 7$. But if we consider $x = 2$, then we find a better solution $(2{,}0{,}1)$, which has sum $S_2 = 3$.
A good work would be to formalize the "rational" algorithm suggested for the two unknowns problem (trying for successive values of x) and find out why it does not work with the three variables problem.

The difficulty of these generalizations, the solutions of which we do not take in consideration here, should help the students realize how easy and fascinating it is to find problems to face, and how research in mathematics could work.

GEOMETRIC PROBLEM: PRESENTATION AND SOLUTION

We do not care about fascinating formulations of this problem either, but again about the possibilities given by the solution. In particular, we will describe how easily a well-driven student can find by himself new problems and attempts to generalize or modify the problem given, having a concrete experience of investigation in mathematics.

DESCRIPTION OF THE PROBLEM

Given a rectangle ABCD, find, if it exists, a rectangle EFGH circumscribed to the previous one, such that the area of the EFGH is twice the area of ABCD. An example of circumscribed rectangles is given by Fig. 2.

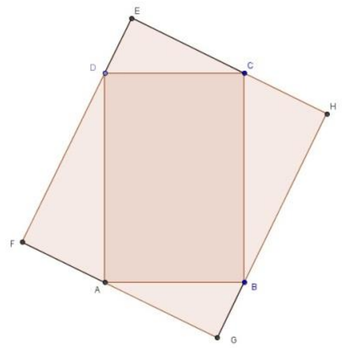

Fig. 2 Example of circumscribed rectangles

A good way to attack the problem is by considering it in GeoGebra. Consult (GeoGebrawiki, *GeoGebra manual*) for some help about the full potential of this tool. Drawing perpendicular lines passing for each point A,B,C,D, we can represent the problem as explained in Fig. 3: the only free parameter is the position of point P:

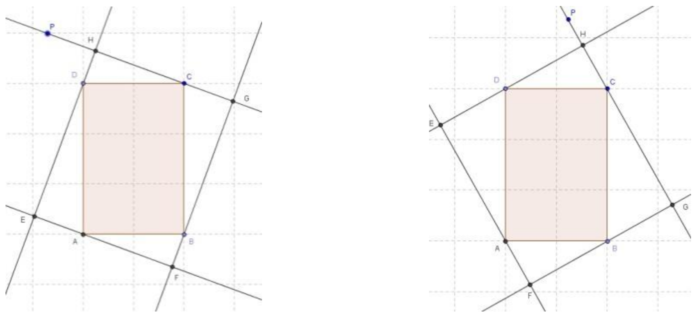

Fig. 3 The problem in GeoGebra, different positions of point P

Through the exploration of the problem via GeoGebra, a student can find a good way to conjecture that a solution always exists, and an hypothesis of how to find it. The main point is to help the student to verify whether his/her intuition is correct or not, by formal demonstration. This way we have a possibility to emphasize the importance of both the intuitive and formal part of mathematics.

The perfect situation arises when some students have “wrong” intuitions (i.e. that are not correct, or that do not lead to the solution) and other students have “right” intuitions (i.e. that are correct and lead to the solution).

A good teacher should analyze deeply any suggest from his/her students, in order to help them comprehend why they are wrong, or to fully comprehend why they are right.

Anyway, we present two possible ways to find a solution.

1) Solution by intuition: with the help of GeoGebra, one could decide to see what happens if the point P is put inside the rectangle ABCD: if P coincides with A, we find the inclination we want, as shown in Fig. 4:

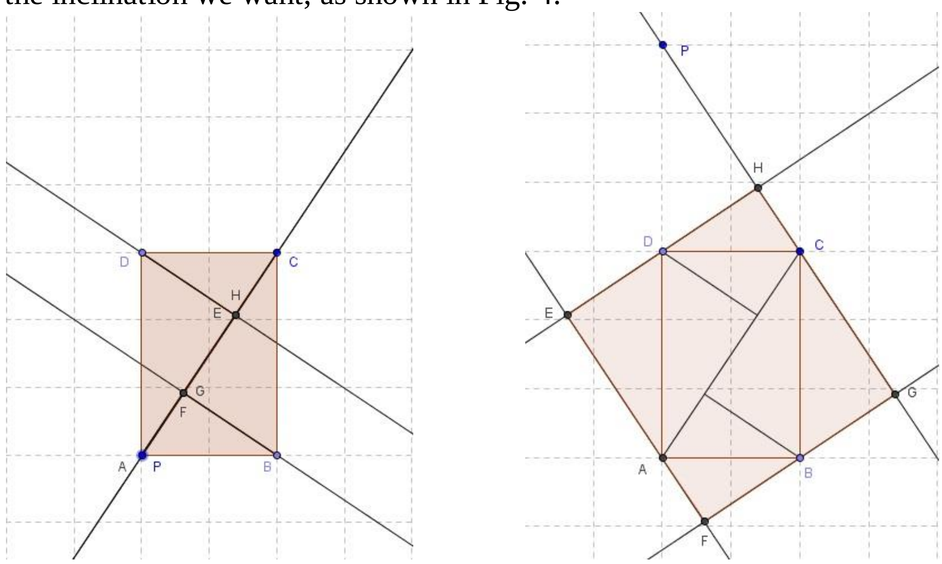

Fig. 4 Idea of the solution to the problem

2) Solution with help of other tools: using GeoGebra at its full potential, we can create a spreadsheet that calculates the area of our rectangles. Moving the point P, we find that at some points the area is very small (even 0), while it easily assumes values greater than double the area of ABCD. One can therefore guess where to put the point P to obtain its solution, and then try to prove it works.

Some problems that may arise (or that should be proposed) during the investigation are here described: